

Thermodynamics of a BTZ black hole solution with an Horndeski source

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In three dimensions, we consider a particular truncation of the Horndeski action that reduces to the Einstein-Hilbert Lagrangian with a cosmological constant Λ and a scalar field whose dynamics is governed by its usual kinetic term together with a nonminimal kinetic coupling. Requiring the radial component of the conserved current to vanish, the solution turns out to be the BTZ black hole geometry with a radial scalar field well-defined at the horizon. This means in particular that the stress tensor associated to the matter source behaves on-shell as an effective cosmological constant term. We construct an Euclidean action whose field equations are consistent with the original ones and such that the constraint on the radial component of the conserved current also appears as a field equation. With the help of this Euclidean action, we derive the mass and the entropy of the solution, and found that they are proportional to the thermodynamical quantities of the BTZ solution by an overall factor that depends on the cosmological constant. The reality condition and the positivity of the mass impose the cosmological constant to be bounded from above as $\Lambda \leq -\frac{1}{l^2}$ where the limiting case $\Lambda = -\frac{1}{l^2}$ reduces to the BTZ solution with a vanishing scalar field. Exploiting a scaling symmetry of the reduced action, we also obtain the usual three-dimensional Smarr formula. In the last section, we extend all these results in higher dimensions where the metric turns out to be the Schwarzschild-AdS spacetime with planar horizon.

I. INTRODUCTION

Since the discovery of the BTZ black hole solution [1], three-dimensional Einstein gravity with a negative cosmological constant has become an important field of investigations. A considerable number of papers has been devoted to the physical and mathematical implications of the BTZ solution in particular in the context of AdS₃/CFT₂ correspondence. Indeed, it is now well accepted that three-dimensional gravity is an excellent laboratory in order to explore and test some of the ideas behind the AdS/CFT correspondence [2].

It is well-known that the static BTZ geometry whose line element is given by

$$ds^2 = -\left(\frac{r^2}{l^2} - M\right)dt^2 + \frac{dr^2}{\frac{r^2}{l^2} - M} + r^2 d\varphi^2, \quad (1)$$

is a solution of the Einstein equations with a fixed value of the negative cosmological constant $-l^{-2}$,

$$G_{\mu\nu} - l^{-2}g_{\mu\nu} = 0.$$

Here, we will exhibit a matter action that sources the BTZ spacetime. In order to achieve this task, the corresponding stress tensor $T_{\mu\nu}$ of the matter source must behave on-shell as an effective cosmological constant term, i. e.

$$T_{\mu\nu}^{\text{on-shell}} = (l^{-2} + \Lambda) g_{\mu\nu}.$$

Indeed, in this case, it is simple to realize that the Einstein equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu},$$

will automatically be satisfied on the BTZ metric (1). For that purpose, we consider the following three-dimensional action

$$S = \int \sqrt{-g} d^3x \left(R - 2\Lambda - \frac{1}{2}(\alpha g_{\mu\nu} - \eta G_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi \right), \quad (2)$$

where R and $G_{\mu\nu}$ stand respectively for the Ricci scalar and the Einstein tensor. This model is part of the so-called Horndeski action which is the most general tensor-scalar action yielding at most to second-order field equations in four dimensions [3]. In three dimensions, the corresponding field equations are also of second order, and are given by

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{1}{2} \left[\alpha T_{\mu\nu}^{(1)} + \eta T_{\mu\nu}^{(2)} \right], \quad (3a)$$

$$\nabla_\mu [(\alpha g^{\mu\nu} - \eta G^{\mu\nu}) \nabla_\nu \phi] = 0, \quad (3b)$$

where the stress tensors $T_{\mu\nu}^{(i)}$ are defined by

$$T_{\mu\nu}^{(1)} = \left(\nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \phi \nabla^\lambda \phi \right). \quad (4)$$

$$T_{\mu\nu}^{(2)} = \frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi R - 2 \nabla_\lambda \phi \nabla_{(\mu} \phi R_{\nu)}^\lambda - \nabla^\lambda \phi \nabla^\rho \phi R_{\mu\lambda\nu\rho} \\ - (\nabla_\mu \nabla^\lambda \phi)(\nabla_\nu \nabla_\lambda \phi) + (\nabla_\mu \nabla_\nu \phi) \square \phi + \frac{1}{2} G_{\mu\nu} (\nabla \phi)^2 \\ - g_{\mu\nu} \left[-\frac{1}{2} (\nabla^\lambda \nabla^\rho \phi)(\nabla_\lambda \nabla_\rho \phi) + \frac{1}{2} (\square \phi)^2 - \nabla_\lambda \phi \nabla_\rho \phi R^{\lambda\rho} \right].$$

The scalar field equation (3b) is a current conservation equation which is a consequence of the shift symmetry of the action, $\phi \rightarrow \phi + \text{const.}$

The first exact black hole solution of these equations without cosmological constant was found in [4]. However, in this case, the scalar field becomes imaginary outside the horizon. Recently, this problem has been circumvented by adding a cosmological constant term yielding to asymptotically locally (A)dS (and even flat for $\alpha = 0$)

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black hole solutions with a real scalar field outside the horizon [5]. The electric charged version of the AdS solutions with a Maxwell field have been studied in [6]. The field equations (3) admit other interesting solutions with a nontrivial and regular time-dependent scalar field on a static and spherically symmetric spacetime [7]. Interestingly enough, this solution in the particular case of $\Lambda = \eta = 0$ reduces to an unexpected stealth configuration on the Schwarzschild metric [7]. There also exist Lifshitz black hole solutions with a time-independent scalar field for a fixed value of the dynamical exponent $z = \frac{1}{3}$, [8]. Solutions for a more general truncation of the Horndeski action that is shift-invariant as well as enjoying the reflection symmetry $\phi \rightarrow -\phi$ have been obtained in [9]. In all these examples, in order to satisfy the radial part of the current conservation (3b) without imposing any constraints on the radial derivative of the scalar field, the geometry has been chosen such that

$$\alpha g^{rr} - \eta G^{rr} = 0. \quad (5)$$

We note that this condition simplifies considerably the field equations in particular in the time-independent case where the full conservation equation (3b) is automatically satisfied without constraining the radial dependence of the scalar field.

In the present work, we will show that the BTZ geometry naturally emerges as a solution of this particular Horndeski action (2) in three dimensions subjected to the condition (5). We will also analyze the thermodynamical implications of such solution and extend all our results in arbitrary dimension. The plan of the paper is organized as follows. In the next section, we present in details the derivation of the solution using the constraint (5). In Sec. III, we construct an Euclidean action whose field equations turn to be consistent with the original ones and such that the constraint on the radial current (5) naturally appears as a field equation. This construction will be useful to obtain the mass and the entropy of the solution. The usual Smarr formula is also derived by exploiting a scaling symmetry of the reduced action. The rotating version of the solution as well as a particular example of time-dependent solution will be reported in Sec. IV. In Sec. V, we extend all the results to arbitrary dimension where the metric solution is nothing but the Schwarzschild-AdS spacetime with planar horizon. The last section is devoted to our conclusions.

II. DERIVATION OF THE SOLUTION

Let us derive the most general solution of the field equations (3) subjected to the condition (5) with an Ansatz of the form

$$\begin{aligned} ds^2 &= -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\varphi^2, \\ \phi &= \phi(r). \end{aligned} \quad (6)$$

For clarity, we define

$$\epsilon_{\mu\nu} := G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} \left[\alpha T_{\mu\nu}^{(1)} + \eta T_{\mu\nu}^{(2)} \right].$$

The condition (5) on the geometry becomes

$$f(r) = \frac{2\alpha r h(r)}{\eta h'(r)}, \quad (7)$$

and automatically implies that the current conservation (3b) is satisfied. Using this last relation, the radial component of the Einstein equations $\epsilon_{rr} = 0$ allows to express the square of the derivative of the scalar field as

$$(\phi')^2 = -\frac{(\alpha + \eta\Lambda)h'}{\alpha^2 r h}.$$

The remaining independent Einstein equation, $\epsilon_{tt} = 0$ or equivalently $\epsilon_{\varphi\varphi} = 0$, yields

$$\epsilon_{tt} \propto (\alpha - \eta\Lambda) [r h'' - h'] = 0.$$

Hence, it is clear that the point defined by $\alpha = \eta\Lambda$ corresponds to a degenerate sector [8], while for $\alpha \neq \eta\Lambda$, the solution is given by

$$h(r) = Cr^2 - M, \quad f(r) = \frac{\alpha}{\eta C} (Cr^2 - M),$$

where C and M are two integration constants. In order to deal with the BTZ metric (1), we choose $C = l^{-2}$ and the coupling constants must be fixed such that

$$\frac{\alpha}{\eta} = l^{-2}. \quad (8)$$

Note that the degenerate sector $\alpha = \eta\Lambda$ corresponds to a choice of the cosmological constant

$$\Lambda^{\text{degenerate}} = l^{-2}. \quad (9)$$

In sum, for $\frac{\alpha}{\eta} = l^{-2}$ and $\Lambda \neq \Lambda^{\text{deg}}$, the solution is given by the BTZ metric (1) together with a radial scalar field

$$\xi(r) := (\phi')^2 = -\frac{2(\Lambda l^2 + 1)}{\eta \left(\frac{r^2}{l^2} - M \right)} \quad (10a)$$

$$\phi(r) = \pm \sqrt{-\frac{2l^2(\Lambda l^2 + 1)}{\eta} \ln \left(\frac{r}{l} + \sqrt{\frac{r^2}{l^2} - M} \right)} \quad (10b)$$

provided that

$$\frac{2l^2(\Lambda l^2 + 1)}{\eta} \leq 0. \quad (11)$$

Various comments can be made concerning this solution. Firstly, we note that for $\Lambda = -l^{-2}$, the scalar field vanishes identically and the solution reduces to the BTZ solution. Secondly, the scalar field is well-defined at the horizon $r_+ = l\sqrt{M}$, and as expected the stress tensor of

the matter part behaves on-shell as an effective cosmological constant

$$\frac{1}{2} \left[\alpha T_{\mu\nu}^{(1)} + \eta T_{\mu\nu}^{(2)} \right]^{\text{on-shell}} = (\Lambda + l^{-2}) g_{\mu\nu}.$$

As a last comment, we remark that for $\alpha > 0$ which corresponds to the right sign of the standard kinetic term, the previous reality conditions (8-11) will imply that $\eta > 0$ and the cosmological constant Λ must be bounded from above as $\Lambda \leq -l^{-2}$. The limiting case $\Lambda = -l^{-2}$ corresponding to the BTZ solution without source.

In what follows, we will derive the mass and the entropy of the solution (10).

III. THERMODYNAMICS OF THE BLACK HOLE SOLUTION

The partition function for a thermodynamical ensemble is identified with the Euclidean path integral in the saddle point approximation around the Euclidean continuation of the classical solution [10]. The Euclidean and Lorentzian action are related by $I_E = -iI$ where the

periodic Euclidean time is $\tau = it$. The Euclidean continuation of the class of metrics considered here is given by [15]

$$ds^2 = N^2(r) F(r) d\tau^2 + \frac{dr^2}{F(r)} + r^2 d\varphi^2.$$

In order to avoid conical singularity at the horizon in the Euclidean metric, the Euclidean time is made periodic with period β and the Hawking temperature T is given by $T = \beta^{-1}$. Since we are only interested in static solution with a radial scalar field, it is enough to consider a *reduced* action principle. However, there is an important subtlety that has to do with the constraint (5) we used in order to derive our solution. Indeed, this constraint together with the fact of looking for a static scalar field make the equation associated to the variation of the scalar field (3b) redundant in the sense that the equation is automatically satisfied. Hence, in our reduced action, the constraint (5) should appear as a field equation in order to deal with an equivalent problem. This can be achieved considering the following Euclidean action

$$I_E := I_E(N, F, \xi) = -2\pi\beta \int_{r_+}^{\infty} N \left[F' + 2\Lambda r + \frac{\alpha}{2} r F \xi + \frac{3}{4} \eta F F' \xi + \frac{\eta}{2} F^2 \xi' \right] dr + B_E, \quad (12)$$

where the dynamical field is chosen to be $\xi(r) := (\phi')^2$ and not the scalar field itself ϕ . Note that r_+ is the location of the horizon and B_E is a boundary term that is fixed by requiring that the Euclidean action has an extremum, that is $\delta I_E = 0$. In this case, the variation with respect to the dynamical fields N, F and ξ yield

$$\begin{aligned} E_N &:= F' + 2\Lambda r + \frac{\alpha}{2} r F \xi + \frac{3}{4} \eta F F' \xi + \frac{\eta}{2} F^2 \xi' = 0, \\ E_F &:= -N' \left(1 + \frac{3}{4} \eta F \xi \right) + N \left(\frac{\alpha}{2} r \xi + \frac{1}{4} \eta F \xi' \right) = 0, \\ E_\xi &:= -\frac{\eta}{2} N' F^2 + N \left(\frac{\alpha}{2} r F - \frac{1}{4} \eta F' F \right) = 0, \end{aligned} \quad (13)$$

and the last equation $E_\xi = 0$ is nothing but the constraint used previously (5) to obtain our solution. At the special point $\alpha = \frac{\eta}{l^2}$, the equations (13) turn out to be equivalent to the original ones supplemented by the constraint (5). Indeed, the most general solution of the system (13) can be derived as follows. For $X(r) := 4 + 3\eta F(r)\xi(r) \neq 0$ [16], we consider the combination

$$-\frac{2\eta F^2}{X} E_F + \frac{\eta N F}{X} E_N + E_\xi = 0,$$

which permits to obtain

$$\xi(r) = -\frac{2(\Lambda l^2 + 1)}{\eta F(r)}.$$

Injecting this expression into $E_N = 0$, one obtains that $F(r) = r^2/l^2 - M$ where M is an integration constant, and finally the equation $E_\xi = 0$ implies that N is constant which can be chosen to 1 without any loss of generality. Hence, at $\alpha = \frac{\eta}{l^2}$, the most general solution of the system (13) is given by

$$N(r) = 1, \quad F(r) = \frac{r^2}{l^2} - M, \quad \xi(r) = -\frac{2(\Lambda l^2 + 1)}{\eta F(r)}, \quad (14)$$

and corresponds to the solution obtained previously (10). We now determine the boundary term of the Euclidean action which is given by

$$\delta B_E = -2\pi\beta \left[\delta F \left(1 + \frac{3}{4} \eta F \xi \right) + \frac{\eta}{2} F^2 \delta \xi \right]_{r_+}^{\infty}. \quad (15)$$

In order to obtain δB_E , we need the variations of the field solutions (14) at infinity

$$\delta F|_{\infty} = -\delta M, \quad (F^2 \delta \xi)|_{\infty} = \frac{2(\Lambda l^2 + 1)}{\eta} \delta F|_{\infty},$$

while at the horizon, they are given by

$$\begin{aligned}\delta F|_{r_+} &= -F'|_{r_+} \delta r_+ = -\frac{4\pi}{\beta} \delta r_+, \\ (F^2 \delta \xi)|_{r_+} &= \frac{2(\Lambda l^2 + 1)}{\eta} \delta F|_{r_+} = -\frac{2(\Lambda l^2 + 1)}{\eta} \frac{4\pi}{\beta} \delta r_+.\end{aligned}$$

Hence, we have

$$I_E = B_E(\infty) - B_E(r_+) = 2\pi \left[\beta M - 4\pi r_+ \right] \left(\frac{1 - \Lambda l^2}{2} \right),$$

and, we can identify the mass \mathcal{M} and the entropy \mathcal{S} of the solution to be given by

$$\mathcal{M} = \frac{\partial I_E}{\partial \beta}, \quad \mathcal{S} = \beta \frac{\partial I_E}{\partial \beta} - I_E,$$

yielding

$$\mathcal{M} = 2\pi M \left(\frac{1 - \Lambda l^2}{2} \right), \quad \mathcal{S} = 8\pi^2 \left(\frac{1 - \Lambda l^2}{2} \right) r_+ \quad (16)$$

Since the Hawking temperature is given by $T = \frac{1}{2\pi} r_+$, it is easy to see that the first law $d\mathcal{M} = T d\mathcal{S}$ holds. In the BTZ case $\Lambda = -l^{-2}$, the scalar field vanishes and the mass and entropy reduce to the thermodynamical quantities of the BTZ solution [1].

We can now go further by exploiting a scaling symmetry of the reduced action in order to obtain the usual three-dimensional Smarr formula in the same lines as those done in Ref. [11]. In fact, it is easy to see that the reduced action (12) enjoys the following scaling symmetry

$$\begin{aligned}\bar{r} &= \sigma r, \quad \bar{N}(\bar{r}) = \sigma^{-2} N(r), \\ \bar{F}(\bar{r}) &= \sigma^2 F(r), \quad \bar{\xi}(\bar{r}) = \sigma^{-2} \xi(r),\end{aligned} \quad (17)$$

from which one can derive a Noether quantity

$$C(r) = N \left[\left(1 + \frac{3}{4} \eta F \xi \right) (-2F + rF') + \frac{\eta}{2} F^2 (r\xi' + 2\xi) \right],$$

which is conserved $C'(r) = 0$ by virtue of the field equations (13). Evaluating this quantity at infinity and at the horizon, one gets

$$C(r = \infty) = M(1 - \Lambda l^2), \quad C(r = r_+) = \frac{4\pi}{\beta} r_+ \left(\frac{1 - \Lambda l^2}{2} \right).$$

Since the Noether charge is conserved, these expressions must be equal

$$M(1 - \Lambda l^2) = \frac{4\pi}{\beta} r_+ \left(\frac{1 - \Lambda l^2}{2} \right),$$

which in turn implies the following Smarr formula

$$\mathcal{M} = \frac{T}{2} \mathcal{S}. \quad (18)$$

This latter corresponds to the standard three-dimensional Smarr formula [12].

IV. ROTATING, TIME DEPENDENT AND STEALTH SOLUTIONS

Operating a Lorentz boost in the plane (t, φ) , we obtain the rotating version of the solution found previously. At the point $\alpha = \eta l^{-2}$, the metric function turns out to be the rotating BTZ

$$ds^2 = -F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 \left(d\varphi - \frac{J}{2r^2} dt \right)^2, \quad (19)$$

where the structural function F is given by

$$F(r) = \frac{r^2}{l^2} - M + \frac{J^2}{4r^2}, \quad (20)$$

and, where J corresponds to the angular momentum. The scalar field solution read

$$\xi(r) := (\phi'(r))^2 = -\frac{2(l^2 \Lambda + 1)}{\eta F(r)}. \quad (21)$$

We also report two solutions with a linear time scalar field on the rotating BTZ metric (19). The first one is obtained for $\alpha = \eta l^{-2}$ and given by

$$\phi(t, r) = q t \pm \int \sqrt{\frac{q^2 \eta - 2F(r)(\Lambda l^2 + 1)}{\eta F(r)^2}} dr, \quad (22)$$

where q is a constant. The second solution corresponds to a stealth configuration, that is a solution where both side of the Einstein equations (3a) vanish identically

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 = \frac{1}{2} \left[\alpha T_{\mu\nu}^{(1)} + \eta T_{\mu\nu}^{(2)} \right]. \quad (23)$$

In fact, for $\Lambda = -1/l^2$, a solution of the stealth equations (23) is given by the rotating BTZ metric (19) together with a time-dependent scalar field

$$\phi(t, r) = q \left(t \pm \int \frac{dr}{F(r)} \right), \quad (24)$$

where q is a constant. This stealth solution is different from the one derived in [13], where in this reference the authors considered as a source a scalar field nonminimally coupled. Moreover, in [13], the time-dependent stealth only exists in the case of the static BTZ metric, that is for $J = 0$.

V. EXTENSION TO HIGHER DIMENSIONS

We now extend our analysis in arbitrary D dimensions with the action

$$S = \int \sqrt{-g} d^D x \left(R - 2\Lambda - \frac{1}{2} (\alpha g_{\mu\nu} - \eta G_{\mu\nu}) \nabla^\mu \phi \nabla^\nu \phi \right) \quad (25)$$

for which the field equations are given by (3). Here, we will consider an Ansatz metric with a planar horizon and a static radial scalar field

$$ds^2 = -N^2 F dt^2 + \frac{dr^2}{F} + r^2 d\vec{x}_{D-2}^2$$

$$\phi = \phi(r).$$

The solution of the field equations for this Ansatz subjected to the constraint (5) is now given by the Schwarzschild-AdS metric with a planar horizon

$$ds^2 = -F(r) dt^2 + \frac{dr^2}{F(r)} + r^2 d\vec{x}_{D-2}^2, \quad (26a)$$

$$F(r) = \frac{r^2}{l^2} - \frac{M}{r^{D-3}}, \quad (26b)$$

$$\xi(r) := (\phi')^2 = -\frac{2(2l^2\Lambda + (D-1)(D-2))}{\eta(D-1)(D-2)F(r)}, \quad (26c)$$

provided that

$$\frac{\alpha}{\eta} = \frac{(D-1)(D-2)}{2l^2}. \quad (27)$$

Note that as in the three-dimensional case, one can obtain an explicit expression of the scalar field

$$\phi(r) = \pm \frac{2}{(D-1)} \sqrt{-\frac{2l^2[2\Lambda l^2 + (D-1)(D-2)]}{\eta(D-1)(D-2)}}$$

$$\times \ln \left[r^{\frac{D-3}{2}} \left(\frac{r}{l} + \sqrt{\frac{r^2}{l^2} - \frac{M}{r^{D-3}}} \right) \right].$$

In the case $D = 3$, the solution reduces to the one previously derived (10) on the BTZ spacetime, and in $D = 4$, this solution was already reported in Ref. [5]. As before, the stress tensor associated to the variation of the matter source behaves on-shell as an effective cosmological constant term, that is

$$\frac{1}{2} \left[\alpha T_{\mu\nu}^{(1)} + \eta T_{\mu\nu}^{(2)} \right]^{\text{on-shell}} = \left(\Lambda + \frac{(D-1)(D-2)}{2l^2} \right) g_{\mu\nu}.$$

The reality condition (26c) together with the relation (27) and requiring the standard kinetic term to have the right sign $\alpha > 0$ impose the cosmological constant Λ to be bounded from above as

$$\Lambda \leq -\frac{(D-1)(D-2)}{2l^2}. \quad (28)$$

The Euclidean action is now given by

$$I_E(N, F, \xi) = \beta \text{Vol}(\Sigma_{D-2}) \int_{r_+}^{\infty} N \left[(D-2)r^{D-3}F' + 2\Lambda r^{D-2} + \frac{\alpha}{2}r^{D-2}F\xi + \frac{3(D-2)}{4}r^{D-3}\eta FF'\xi \right. \\ \left. + \frac{(D-2)\eta}{2}r^{D-3}F^2\xi' + (D-2)(D-3) \left(r^{D-4}F + \frac{\eta}{4}F^2\xi r^{D-4} \right) \right] dr + B_E, \quad (29)$$

where $\text{Vol}(\Sigma_{D-2})$ stands for the volume of the compact $(D-2)$ -dimensional planar manifold, and $r_+ = (l^2 M)^{1/(d-1)}$ is the location of the horizon. The variation with respect to the dynamical fields N, F and ξ yield

$$E_N := (D-2)r^{D-3}F' + 2\Lambda r^{D-2} + \frac{\alpha}{2}r^{D-2}F\xi$$

$$+ \frac{3(D-2)}{4}r^{D-3}\eta FF'\xi + \frac{(D-2)\eta}{2}r^{D-3}F^2\xi'$$

$$+ (D-2)(D-3) \left(r^{D-4}F + \frac{\eta}{4}F^2\xi r^{D-4} \right) = 0,$$

$$E_F := -N' \left((D-2)r^{D-3} + \frac{3}{4}(D-2)r^{D-3}\eta F\xi \right)$$

$$+ N \left(\frac{\alpha}{2}r^{D-2}\xi + \frac{(D-2)}{4}r^{D-3}\eta F\xi' \right.$$

$$\left. - \frac{(D-3)(D-2)}{4}r^{D-4}F\eta\xi \right) = 0,$$

$$E_\xi := -\frac{(D-2)}{2}\eta r^{D-3}N'F^2 +$$

$$N \left(\frac{\alpha}{2}r^{D-2}F - \frac{(D-2)}{4}r^{D-3}\eta FF' \right.$$

$$\left. - \frac{(D-3)(D-2)}{4}r^{D-4}F^2\eta \right) = 0,$$

and the last equation $E_\xi = 0$ is again proportional to the constraint (5) used previously to obtain the solution. As before, at the special point (27), this system of equations is equivalent to our original equations supplemented by the constraint (5) which also appears as a field equation. The most general solution yields to (26) together with $N(r) = 1$.

We are now in position to compute the variation

$$\delta B_E = -\beta \text{Vol}(\Sigma_{D-2})(D-2)r^{D-3} \times \left[\delta F \left(1 + \frac{3}{4}\eta F \xi \right) + \frac{\eta}{2} F^2 \delta \xi \right]_{r_+}^\infty, \quad (30)$$

which permits to obtain that

$$I_E = \frac{(D-1)(D-2) - 2l^2\Lambda}{2(D-1)} [\beta M - 4\pi r_+^{D-2}],$$

We derive the mass $\mathcal{M} = \frac{\partial I_E}{\partial \beta}$ and the entropy $\mathcal{S} = \beta \frac{\partial I_E}{\partial \beta} - I_E$, that read

$$\mathcal{M} = \left[\frac{(D-1)(D-2) - 2l^2\Lambda}{2(D-1)} \right] M \text{Vol}(\Sigma_{D-2}) \quad (31)$$

$$\mathcal{S} = \left[\frac{(D-1)(D-2) - 2l^2\Lambda}{2(D-1)} \right] 4\pi r_+^{D-2} \text{Vol}(\Sigma_{D-2}),$$

and once again, one can easily check that the first law holds. For the Schwarzschild-AdS case, that is for $\Lambda = -\frac{(D-1)(D-2)}{2l^2}$, these formula reduces to those found in [14]. Finally, the Noether conserved quantity

$$C(r) = N r^{D-3} (D-2) \left[\left(1 + \frac{3}{4}\eta F \xi \right) (-2F + rF') + \frac{\eta}{2} F^2 (r\xi' + 2\xi) \right], \quad (32)$$

which is a consequence of the scaling symmetry of the reduced action (29)

$$\bar{r} = \sigma r, \quad \bar{N}(\bar{r}) = \sigma^{1-D} N(r),$$

$$\bar{F}(\bar{r}) = \sigma^2 F(r), \quad \bar{\xi}(\bar{r}) = \sigma^{-2} \xi(r), \quad (33)$$

permits to derive the following Smarr formula

$$\mathcal{M} = \frac{1}{D-1} T \mathcal{S}. \quad (34)$$

One may mention that the previous scaling symmetry will not be possible in the case of spherical or hyperboloid horizon.

VI. CONCLUSIONS

We have been concerned with a particular truncation of the Horndeski theory in three dimensions given by the Einstein-Hilbert piece plus a cosmological constant and a scalar field with its usual kinetic term and a nonminimal kinetic coupling. For this model, we have derived

the most general solution subjected to the condition (5). In this case, the metric turns to be the BTZ spacetime and the radial scalar field is shown to be well-defined at the horizon. We have seen that such solution occurs because the stress tensor associated to the variation of the matter source behaves on-shell as a cosmological constant term. The constraint on the radial component of the conserved current (5) together with the fact of looking for a static scalar field only impose a restriction on the geometry and not on the scalar field. In other words, this means that the field equation associated to the variation of the scalar field is automatically satisfied without imposing any restriction on the radial dependence of the scalar field. In order to compute the mass and the entropy of the solution, we have constructed an Euclidean action whose field equations turn out to be equivalent to the original Einstein equations and such that the constraint on the radial component of the conserved current appears also as a field equation. This last fact has resulted to be primordial to obtain the mass and the entropy, and we have verified that the first law was satisfied. This reduced action has also be useful in order to derive the usual Smarr formula by exploiting a scaling symmetry. We have extended all these results in arbitrary dimension where the metric solution is nothing but the Schwarzschild-AdS spacetime with a planar horizon. In this case also, we have been able to construct an Euclidean action whose field equations are equivalent to the original ones supplemented by the constraint condition on the geometry (5). In all these examples, the horizon topology is planar but this hypothesis is only essential in order to establish a scaling symmetry of the reduced action. In fact, in the spherical or hyperboloid cases, one would be able to construct along the same lines the Euclidean action sharing the same features except enjoying the scaling symmetry. In higher dimensions, the authors of Ref. [5] have considered the same model and found black hole solutions with spherical and hyperboloid horizon topology. In these cases, they have computed the thermodynamical quantities by regularizing the action with the use of a regular soliton solution. It will be interesting to see wether the Euclidean approach described here may yield the same results. It is also appealing that up to now all the known solutions of the equations (3) are those where the constraint (5) is imposed. It will be interesting to see wether there exist other solutions for which the radial component of the current conservation is not vanishing. Finally, we have also obtained a particular time-dependent solution where the scalar field depends linearly on the time. For such solution, the issue concerning the thermodynamical analysis is not clear for us. Hence, a natural work will consist in providing a consistent Hamiltonian formalism in order to compute the mass, the entropy and also to give a physical interpretation on the additional constant q that appears in the solution. This problem will also be relevant in the context of the Lifshitz case where the known solutions [8, 9] are necessarily time-dependent.

Acknowledgments

We thank Julio Oliva for useful discussions. MB is supported by BECA DOCTORAL CONICYT 21120271.

MH is partially supported by grant 1130423 from FONDECYT and from CONICYT, Departamento de Relaciones Internacionales “Programa Regional MATH-AMSUD 13 MATH-05”.

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 - [15] For the Ansatz considered in (6), this will correspond to $h(r) = N(r)^2 F(r)$ and $f(r) = F(r)$.
 - [16] For $X(r) = 0$, one ends with a particular case of (10) with a fixed value of the cosmological constant $\Lambda = -\frac{1}{3l^2}$.